Module 2: First-Order Partial Differential Equations

The mathematical formulations of many problems in science and engineering reduce to study of first-order PDEs. For instance, the study of first-order PDEs arise in gas flow problems, traffic flow problems, phenomenon of shock waves, the motion of wave fronts, Hamilton-Jacobi theory, nonlinear continum mechanics and quantum mechanics etc.. It is therefore essential to study the theory of first-order PDEs and the nature their solutions to analyze the related physical problems.

In Module 2, we shall study first-order linear, quasi-linear and nonlinear PDEs and methods of solving these equations. An important method of characteristics is explained for these equations in which solving PDE reduces to solving an ODE system along a characteristics curve. Further, the Charpit's method and the Jacobi's method for nonlinear first-order PDEs are discussed.

This module consists of seven lectures. Lecture 1 introduces some basic concepts of first-order PDEs such as formulation of PDEs, classification of first-order PDEs and Cauchy's problem for first-order PDEs. In Lecture 2, we study first-order linear PDEs and the parametric form of solution of first-order PDEs. In Lecture 3, we study a first-order quasi-linear PDE and discuss the method of characteristics for a first-order quasi-linear PDE. Lecture 4 is devoted to nonlinear first-order PDEs and Cauchy's method of characteristics for finding solutions of these equations. Lecture 5 is focused on the compatible system of equations and Charpit's method for solving nonlinear equations. In Lecture 6, we consider some special type of PDEs and method of obtaining their general integrals. Finally, the Jacobi's method for solving nonlinear PDEs is discussed in Lecture 7.

Lecture 1 First-Order Partial Differential Equations

A first order PDE in two independent variables x, y and the dependent variable z can be written in the form

$$f(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0. \tag{1}$$

For convenience, we set

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

Equation (1) then takes the form

$$f(x, y, z, p, q) = 0. \tag{2}$$

The equations of the type (2) arise in many applications in geometry and physics. For instance, consider the following geometrical problem.

Example 1. *Find all functions $z(x, y)$ such that the tangent plane to the graph $z = z(x, y)$ at any arbitrary point $(x_0, y_0, z(x_0, y_0))$ passes through the origin characterized by the PDE $xz_x + yz_y - z = 0$.*

The equation of the tangent plane to the graph at $(x_0, y_0, z(x_0, y_0))$ is

$$z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0) - (z - z(x_0, y_0)) = 0.$$

This plane passes through the origin $(0, 0, 0)$ and hence, we must have

$$-z_x(x_0, y_0)x_0 - z_y(x_0, y_0)y_0 + z(x_0, y_0) = 0. \tag{3}$$

For the equation (3) to hold for all (x_0, y_0) in the domain of z, z must satisfy

$$xz_x + yz_y - z = 0,$$

which is a first-order PDE.

Example 2. *The set of all spheres with centers on the z-axis is characterized by the first-order PDE $yp - xq = 0$.*

The equation

$$x^2 + y^2 + (z - c)^2 = r^2, \tag{4}$$

where r and c are arbitrary constants, represents the set of all spheres whose centers lie on the z-axis. Differentiating (4) with respect to x, we obtain

$$2\left(x + (z - c)\frac{\partial z}{\partial x}\right) = 2\left(x + (z - c)p\right) = 0. \tag{5}$$

Differentiate (4) with respect to y to have

$$y + (z - c)q = 0. \tag{6}$$

Eliminating the arbitrary constant c from (5) and (6), we obtain the first-order PDE

$$yp - xq = 0. \tag{7}$$

Equation (4) in some sense characterized the first-order PDE (7).

Example 3. *Consider all surfaces described by an equation of the form*

$$z = f(x^2 + y^2), \tag{8}$$

where f is an arbitrary function, described by the first-order PDE.

Writing $u = x^2 + y^2$ and differentiating (8) with respect to x and y, it follows that

$$p = 2xf'(u); \qquad q = 2yf'(u),$$

where $f'(u) = \frac{df}{du}$. Eliminating $f'(u)$ from the above two equations, we obtain the same first-order PDE as in (7).

Remark 4. *The function z described by each of the equations (4) and (8), in some sense, a solution to the PDE (7). Observe that, in Example 2, PDE (7) is formulated by eliminating arbitrary constants from (4) whereas in Example 3, PDE (7) is formed by eliminating an arbitrary function.*

1 Formation of first-order PDEs

The applications of conservation principles often yield a first-order PDEs. We have seen in the previous two examples that a first-order PDE can be formed either by eliminating arbitrary constants or an arbitrary function involved. Below, we now generalize the arguments of Example 2 and Example 3 to show that how a first-order PDE can be formed.

Method I (*Eliminating arbitrary constants*): Consider two parameters family of surfaces described by the equation

$$F(x, y, z, a, b) = 0, \tag{9}$$

where a and b are arbitrary constants. Equation (9) may be thought of as a generalization of the relation (4).

Differentiating (9) with respect to x and y, we obtain

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \tag{10}$$

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0. \tag{11}$$

Eliminate the constants a, b from equations (9), (10) and (11) to obtain a first-order PDE of the form

$$f(x, y, z, p, q) = 0. \tag{12}$$

This shows that a family of surfaces described by the relation (9) gives rise to a first-order PDE (12).

Method II (*Eliminating arbitrary function*): Now consider the generalization of Example 3. Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two known functions of x, y and z satisfying a relation of the form

$$F(u, v) = 0, \tag{13}$$

where F is an arbitrary function of u and v. Differentiating (13) with respect to x and y lead to the equations

$$F_u(u_x + u_z p) + F_v(v_x + v_z p) = 0$$

$$F_u(u_y + u_z q) + F_v(v_y + v_z q) = 0.$$

Eliminating F_u and F_v from the above two equations, we obtain

$$p\frac{\partial(u, v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}, \tag{14}$$

which is a first-order PDE of the form $f(x, y, z, p, q) = 0$. Here, $\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x$.

2 Classification of first-order PDEs

We classify the equation (1) depending on the special forms of the function f. If (1) is of the form

$$\boxed{a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} + c(x, y)z = d(x, y)}$$

then it is called **linear** first-order PDE. Note that the function f is linear in $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and z with all coefficients depending on the independent variables x and y only.

If (1) has the form

$$\boxed{a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} = c(x, y, z)}$$

then it is called **semilinear** because it is linear in the leading (highest-order) terms $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. However, it need not be linear in z. Note that the coefficients of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are functions of the independent variables only.

If (1) has the form

$$\boxed{a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z)}$$

then it is called **quasi-linear** PDE. Here the function f is linear in the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with the coefficients a, b and c depending on the independent variables x and y as well as on the unknown z. Note that linear and semilinear equations are special cases of quasi-linear equations.

Any equation that does not fit into one of these forms is called **nonlinear**.

EXAMPLE 5.

1. $xz_x + yz_y = z$ (linear)
2. $xz_x + yz_y = z^2$ (semilinear)
3. $z_x + (x+y)z_y = xy$ (linear)
4. $zz_x + z_y = 0$ (quasilinear)
5. $xz_x^2 + yz_y^2 = 2$ (nonlinear)

3 Cauchy's problem or IVP for first-order PDEs

Recall the initial value problem for a first-order ODE which ask for a solution of the equation that takes a given value at a given point of $\mathbb{R}$. The IVP for first-order PDE ask for a solution of (2) which has given values on a curve in $\mathbb{R}^2$. The conditions to be satisfied in the case of IVP for first-order PDE are formulated in the classic problem of Cauchy which may be stated as follows:

Let C be a given curve in $\mathbb{R}^2$ described parametrically by the equations

$$x = x_0(s), \quad y = y_0(s); \quad s \in I, \tag{15}$$

where $x_0(s)$, $y_0(s)$ are in $C^1(I)$. Let $z_0(s)$ be a given function in $C^1(I)$. The IVP or Cauchy's problem for first-order PDE

$$f(x,y,z,p,q) = 0 \tag{16}$$

is to find a function $u = u(x,y)$ with the following properties:

- $u(x, y)$ and its partial derivatives with respect to x and y are continuous in a region Ω of $\mathbb{R}^2$ containing the curve C.
- $u = u(x, y)$ is a solution of (16) in Ω, i.e.,

$$f(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0 \quad \text{in } \Omega.$$

- On the curve C

$$u(x_0(s), y_0(s)) = z_0(s), \quad s \in I. \tag{17}$$

The curve C is called the initial curve of the problem and the function $z_0(s)$ is called the initial data. Equation (17) is called the initial condition of the problem.

NOTE: Geometrically, Cauchy's problem may be interpreted as follows: To find a solution surface $u = u(x, y)$ of (16) which passes through the curve C whose parametric equations are

$$x = x_0(s), \quad y = y_0(s) \quad z = z_0(s). \tag{18}$$

Further, at every point of which the direction $(p, q, -1)$ of the normal is such that

$$f(x, y, z, p, q) = 0.$$

The proof of existence of a solution of (16) passing through a curve with equations (18) requires some more assumptions on the function f and the nature of the curve C. We now state the classic theorem due to Kowalewski in the following theorem (cf. [10]).

THEOREM 6. *(Kowalewski) If $g(y)$ and all its derivatives are continuous for $|y - y_0| < \delta$, if x_0 is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$, and if $f(x, y, z, q)$ and all its partial derivatives are continuous in a region S defined by*

$$|x - x_0| < \delta, \quad |y - y_0| < \delta, \quad |q - q_0| < \delta,$$

then there exists a unique function $\phi(x, y)$ such that:
(a) $\phi(x, y)$ and all its partial derivatives are continuous in a region
$\Omega : |x - x_0| < \delta_1, \ \ |y - y_0| < \delta_2$;
(b) For all (x, y) in Ω, $z = \phi(x, y)$ is a solution of the equation

$$\frac{\partial z}{\partial x} = f(x, y, z, \frac{\partial z}{\partial y})$$

(c) For all values of y in the interval $|y - y_0| < \delta_1$, $\phi(x_0, y) = g(y)$.

We conclude this lecture by introducing different kinds of solutions of first-order PDE.

Definition 7. *(**A complete solution or a complete integral**) Any relation of the form*

$$\boxed{F(x, y, z, a, b) = 0} \tag{19}$$

*which contains two arbitrary constants a and b and is a solution of a first-order PDE is called **a complete solution or a complete integral** of that first-order PDE.*

Definition 8. *(**A general solution or a general integral**) Any relation of the form*

$$\boxed{F(u, v) = 0}$$

involving an arbitrary function F connecting two known functions $u(x, y, z)$ and $v(x, y, z)$ and providing a solution of a first-order PDE is called a general solution or a general integral of that first-order PDE.

It is possible to derive a general integral of the PDE once a complete integral is known.

With $b = \phi(a)$, if we take any one-parameter subsystem

$$f(x, y, z, a, \phi(a)) = 0$$

of the system (19) and form its envelope, we obtain a solution of equation (16). When $\phi(a)$ is arbitrary, the solution obtained is called the general integral of (16) corresponding to the complete integral (19).

When a definite $\phi(a)$ is used, we obtain a particular solution.

Definition 9. *(**A singular integral**) The envelope of the two-parameter system (19) is also a solution of the equation (16). It is called the singular integral or singular solution of the equation.*

NOTE: The general solution of an equation of type (1) can be obtained by solving systems of ODEs. This is not true for higher-order equations or for systems of first-order equations.

Practice Problems

1. Classify whether the following PDE is linear, quasi-linear or nonlinear:

$$(a)\ zz_x - 2xyz_y = 0;\ \ (b)\ z_x^2 + zz_y = 2;\ \ (c)\ z_x + 2z_y = 5z;\ \ (d)\ xz_x + yz_y = z^2.$$

2. Eliminate the arbitrary constants a and b from the following equations to form the PDE:

$$(a)\ ax^2 + by^2 + z^2 = 1;\ \ (b)\ z = (x^2 + a)(y^2 + b).$$

3. Show that $z = f(xy)$, where f is an arbitrary differentiable function satisfies

$$xz_x - yz_y = 0,$$

and hence, verify that the functions $\sin(xy)$, $\cos(xy)$, $\log(xy)$ and e^{xy} are solutions.

4. Eliminate the arbitrary function f from the following and form the PDE:

$$(a)\ z = x + y + f(xy); \quad (b)\ z = f(\frac{xy}{z}).$$